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REVERSE ARITHMETIC OPERATIONS AND MENTAL CALCULUS IN DIGITAL PLAY MATH FARM

OPERACIONES ARITMÉTICAS INVERSAS Y CÁLCULO MENTAL EN EL JUEGO DIGITAL GRANJA DE MATH

OPERAÇÕES ARITMÉTICAS INVERSAS E CÁLCULO MENTAL NO JOGO DIGITAL FAZENDINHA MATEMÁTICA

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Abstract
Based on the concepts of reversibility and mental calculus and believing in the potential of digital games for mathematical learning, we report, in this article, an experience developed with students from the Final Years of Elementary School, when they played and evaluated a digital educational game of mathematical content called Fazendinha Matemática. Within a qualitative approach we analyze the experience obtaining as results that the game allowed to explore and stimulate the perception of students about the relationships that the multiplication and division operations have one as inverse of the other; stimulate mental calculus; make them identify the mathematics present in the game; evaluate their interest in the game and suggest changes to make it even more attractive. We conclude that one way to help students understand inverse operations and, therefore, the concept of reversibility, is to insert them into rich environments, exploring mental calculus and making use of didactic resources, as is the case of educational typing games.

Keywords: Digital educational games; Mental calculus; Mathematics education; Reversibility; Inverse operations.

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Resumen
Basado en el concepto de reversibilidade y cálculo mental y creyendo en el potencial de los juegos digitales para el aprendizaje matemático, informamos, en este artículo, de una experiencia desarrollada con los estudiantes de la Escuela Primaria, cuando jugaron y evaluaron un juego educativo digital de contenido matemático llamado Granja de Math. Dentro de un enfoque cualitativo analizamos la experiencia obteniendo como resultados que el juego permitió explorar y estimular la percepción de los estudiantes sobre las relaciones que las operaciones de multiplicación y división tienen una como inversa de la otra; estimular el cálculo mental; hacer que identifiquen las matemáticas presentes en el juego; evaluar su interés en el juego y sugerir cambios para hacerlo aún más atractivo. Concluimos que una manera de ayudar a los estudiantes a entender las operaciones inversas y, por lo tanto, el concepto de reversibilidad, es involucrarlos en entornos ricos, explorando el cálculo mental y haciendo uso de los recursos didácticos, como es el caso de los juegos digitales educativos.

Palabras clave: Juegos educativos digitales; Cálculo mental; Educación matemática; Reversibilidad; Operaciones inversas.

Resumo
Baseando-se nos conceitos de reversibilidade e cálculo mental e acreditando no potencial dos jogos digitais para a aprendizagem matemática, relatamos, neste artigo, uma experiência desenvolvida com estudantes dos Anos Finais do Ensino Fundamental, quando estes jogaram e avaliaram um jogo digital educacional de conteúdo matemático chamado Fazendinha Matemática. Dentro de uma abordagem qualitativa analisamos a experiência obtendo como resultados que o jogo permitiu explorar e estimular a percepção dos alunos sobre as relações que as operações de multiplicação e divisão possuem uma como inversa da outra; estimular o cálculo mental; fazê-los identificar a matemática presente no jogo; avaliar o interesse deles pelo jogo e sugerir mudanças para torná-lo ainda mais atrativo. Concluímos que uma maneira de ajudar os alunos a compreender as operações inversas e, portanto, o conceito de reversibilidade, é inseri-los em ambientes ricos, explorando o cálculo mental e fazendo uso de recursos didáticos, como é o caso dos jogos digitais educacionais.

Palavras-chave: Jogos educacionais digitais; Educação matemática; Reversibilidade; Operações inversas.

Reversible thinking in arithmetic operations and digital educational games

There are not a few students who complete the first stage of elementary education early years without understanding the fundamental operations of mathematics. In addition to handling techniques to add, subtract, multiply, and divide, students should understand the relationships between these operations and know how to use them to discover other mathematical ideas and concepts. We can affirm that without difficulties, many students apply operative techniques to solve multiplication and division. However, few are the ones who recognize the relationships between these operations.

It is common to say that there are four operations, which are addition, subtraction, multiplication, and division. However, there are only two independent operations: addition and multiplication, both operating in different ways. Subtraction does not exist as an independent operation, it is a particular case of complementary addition (for example, given a = b + c, the relation b = ac can be introduced by asking: what is missing in "c" to be equivalent to "a"). Also, the division does not exist as an independent operation, it is the inverse operation of multiplication. The understanding of these relationships does not seem so obvious and goes through the understanding of concepts such as equivalence and, in which reversibility will be the object of our attention in this article.

The concept of *reversibility* goes back to Piaget's studies, specifically when he presents the stages of cognitive development (sensorimotor, preoperative, concrete operation, and formal operation). In fact, "the central importance in Piaget's theory is the idea that there are discrete stages of development, each with its properties and characteristics" (CASTRO; RICO; CASTRO, 1996, p.62) and ranging from birth to maturity. According to these authors, since each stage has a different way of thinking we can say that each of them has a different arithmetic thought.

For Inhelder and Piaget (1958), reversibility begins from the concrete operative stage, when the main characteristic of the child is to think logically on the operations he performs and having flexible thinking, starting to pay attention to other aspects and not only one. Reversibility can be understood as a cognitive skill that allows us to carry out an inverse (compensatory) action to cancel previous transformations and return to the starting point. These authors considered reversibility as a fundamental requirement in a series of problems in mathematics.

Studies in the Mathematical Education area have shown reversibility in many mathematical operations, fractions, algebra, derivative and integral calculations, among others, in which we often undo and redo mathematical processes (HACIOMEROGLU; ASPINWALL; PRESMEG, 2009; RAMFUL, 2008; MAFULAH; JUNIATI; SISWONO, 2017).

Krutetskii's (1976) work is a reference to reversibility. Studying gifted students, he identified several mathematical skills related to success in problem-solving, including reversibility and flexibility. Flexibility would be the ability to think in different ways to find different solutions, without too many difficulties, in a flexible way. (HACIOMEROGLU; ASPINWALL; PRESMEG, 2009)

For Krutetskii (1976) unilateral thinking in mathematics, one that moves in one direction can be an obstacle to the use of other modes of thought and points out that reversibility would be the opposite. It would be the ability of thought to establish two-way relationships. “In an inverse line of thought, the thought does not always have to travel precisely the same path, but simply moves in reverse order” (KRUTETSKII, 1976, p.287, apud HACIOMEROGLU; ASPINWALL; PRESMEG, 2009).

For Wierlewski (2005) reversibility

> It is the establishment of two modes of associations of the form A ↔ B as opposed to a mode of connections of the form A → B, with function only in one direction. It is the reversibility of the mental process in reasoning, that is, thinking in the opposite direction of the result. In an inverse chain, the thought does not always have to follow the same route but must move in reverse order. If the direction of the initial thought is A to F, it now moves in the direction of F to A; however, all connections and the sequence of associations do not necessarily have to occur in the strictly reverse order. The intermediary connections can differ and implies that the specific path that thought takes may also differ. In this case, an inverse chain cannot always be reduced to inverse associations. (WIELEWSKI, 2005, p.63-4)

Thus, reversibility means the reconstruction of a mental process to change a direct line of thought to reverse thinking. (KRUTETSKII, 1968 apud WIELEWSKI, 2005).

When resuming with the arithmetic operations, we observe the usual practice that these operations must be worked linearly, first the addition and finally the division and, this ends up limiting that they can be presented concurrently and to do a more immediate work to show the reverse side of operations.

Students' difficulties in the context of arithmetic operations can be explained by several factors such as didactic, content, or others. The fact is that there is a lack of work more focused on understanding the inverse relationships between these operations.

Mental calculation in arithmetic operations

Our approach for mental calculation in this work is justified for two reasons: the first, for admitting this as a cognitive skill, which like reversibility, will require from the individual the ability to undo and redo mental processes, to exercise flexibility of thinking; the second reason is that the experience carried out in this research requires the student to be resourceful in this type of calculation.

Belmonte (2003) in his work defines the mental calculation as the thought calculation and the written calculation to refer to the automatic calculation, and presents the following table with the main differences between them:

Table: Differences between written calculation and mental calculation

Written calculation	Mental calculation (thought)
It is in general. It usually uses a single technique for any pair of numbers. Its operation is always homogeneous. It is the same for different individuals.	There are many techniques, in which you have to select the one that best suits each particular situation. It even varies from one individual to another.
It uses numerical properties implicitly.	Make explicit and conscious use of the numerical properties you need on each occasion.
The repertoire is pre-fixed and limited. Its use is memorable.	The amount of available repertoire plays a very important role. A good structure allows for better mobilization.
Errors are difficult to detect and correct.	You always have a conscious watch over the error.
It is reassuring, reliable.	It creates restlessness and is fast.
When it is exercised, it produces many errors.	When it is not exercised, it is not produced.

Source: Belmonte (2003, p.179)

Although the author makes this difference, he points out that these two calculations are complementary and that the work with mental calculation in teaching has its didactic importance because it expresses the properties of the operations, using the relationship between the different operations, causes situations of approximation and estimation, helps to mobilize and structure results, can be treated as an open problem and favors the conscious evolution of calculation strategies. He highlights “mental calculation deepens the knowledge of numbers, numbering systems, the meaning of operations, and the relationships between them and the calculation algorithms” (BELMONTE, 2003, p.181-2).

Belmonte (2003) observes that in the execution of a mental calculation technique, it is necessary, for example, to use properties of operations, as in the case of finding the product 45 x 18 in which associative and commutative properties are used. To solve the 44 x 25 product, it may be useful to transform one number into another to facilitate the calculation, and depending on the type of transformation may require knowledge of the relationship between multiplication and division. (BELMONTE, 2003, p.181-2)

According to Gómez (1998), the mental calculation is characterized by being head-on, being done quickly, based on a limited set of numerical facts, and requires skills such as counting, compensations, decompositions, and other transformations.

The mental calculation requires concentration, attention, and interest, and they are determining factors for achieving success in results (GÓMEZ, 1998). This author distinguishes between mental stimulus-response and mental calculus. He mentions the multiplication tables as an example and the second mentions personal reflections rarely worked on at school.

In this context, we start from the problem of the students' understanding of the concept of reversibility, of inverse operations, involving a work dedicated to stimulating mental calculation and the use of didactic resources with the capacity to arouse students' interest towards learning, as in the case with digital educational games.

Digital educational games as learning tools

As a didactic tool, the games help in the teaching and learning process, winning over teachers and students for their playful side and promoting more pleasant relationships in the school environment. Students enjoy the presence of games, have fun, and learn while playing. Alves and Biachin (2010, p.283) understand that "the game as a learning tool is a resource of extreme interest to educators since its importance is directly linked to the development of human beings in a social, creative, affective, historical and cultural perspective".

For Kishimoto (1998, p.19) material or activity can be considered an educational game when it has a playful and educational role at the same time. In their playful role, games arouse the interest of the students, are fun, and have pleasurable effects. In their educational role, they can provide both the fixation and the learning of concepts and skills, to stimulate and expand thinking and response strategies, because "when playing, the child experiences, invents, discovers, learns and confers skills. [...] the game is important, not only to encourage imagination in children but also to assist in the development of social and cognitive skills" (ALVES; BIACHIN, 2010, p. 283).

In the case of digital games, we understand that the games are governed by a computer program. In the case of digital educational games, the games are seen as "a new class of learning objects as a bridge between students [...] and the content to be studied.

[...] a more dynamic way of learning, in which the student is the protagonist of his learning, following his own pace”. (DIAS; GUSMÃO; MARQUES, 2017)

Dias, Gusmão, and Marques (2017) observed that digital educational games can increase students' educational experience as they are already familiar with these resources.

Bomfoco and Azevedo (2012) observed that the player is always able to learn while playing and they said that in the studies by Gee (2004), systemic thinking is among the good learning principles that digital games can bring, encouraging players to think about the relationships between events, facts, and skills in them.

According to Telles (2015), the use of technologies such as video games has contributed to cognitive development, as in the case with learning languages, interfaces, and software, and that in the educational field many researchers have endeavored to study relationships between cognition, learning, and electronic games. One of these authors has been Lynn Alves, who has been developing research in Bahia, articulating digital games and learning from its users, in front of his research group Virtual Communities with one of its objectives “to make the school learning space more attentive to languages that seduce students immersed in a digital culture” (ALVES, 2008, p.6).

For Alves (2008), we can create a space in schools for teachers to get in touch with the proposal of games and identify ethical, political, ideological, cultural issues in them, among others that can be discussed with their students, mediating the construction of new meanings for the narratives proposed in the games. Thus, the author understands that digital games are

> [...] cultural phenomena that require the construction of different perspectives, going beyond Manichaean perspectives as if these cultural elements were always the bad guys in the stories that involve violent behaviors, physical inactivity, long hours of interaction with games, school demotivation, failure, and dropping out of school. These uncritical readings, built from a reductionist point of view, restrict the possibilities for dialogue between teachers, gamers/students, and the universe of games. (ALVES, 2008, p.8)

Studying the contributions of the use of electronic cognitive games to the development of cognitive skills, Ramos et al (2015) observed that there is an important association between these variables, highlighting attention skill, and attention training can increase the reasoning capacity.

In all this context, although the scientific scope has no answers to say in fact how the individual understands, how the student understands a mathematical object, such as the concept of inverse operation, we consider opportune researches aimed at looking at other aspects, such as the use of digital educational games to improve learning and understanding of concepts.

Materials and Methods

In this work, we adopted a qualitative approach because it fits the educational needs, as it is research about the action (CHIZZOTTI, 2006) and rich in descriptive details (BOGDAN; BIKLEN, 1994). It is exploratory research as it has a "[...] triple purpose: to develop hypotheses, to increase the researcher's familiarity with the environment, fact or phenomenon, to carry out more precise future research, or to modify and clarify concepts" (MARCONI; LAKATOS, 2010, P. 171). Thus, we will present the results of a digital game of mathematical content, tested with elementary school students to verify the potential of the game to stimulate and develop reversible thinking in arithmetic operations and to stimulate mental calculation.

The game is the result of studies carried out in the research projects "Development and application of video games to enhance the teaching and learning of Mathematics in Basic Education" (GUSMÃO, 2013) and "Didactic sequences for increasing mathematical cognition and metacognition of students from initial years of elementary school" (GUSMÃO, 2009), linked to the Department of Exact and Technological Sciences (DCET) of the *Universidade Estadual do Sudoeste da Bahia* (UESB).

The initial idea was to apply the game to students from the same public school where the analog sequence "*Fazendinha Matemática*" had already been applied. However, the school was deactivated due to the municipalization of elementary education and due to a series of obstacles by other public schools that had their replacement schedule due to strikes. Thus, we opted to apply to a private school, middle and upper class, who kindly showed concern and welcomed us. At this school, the game was tested in two classes, with 34 students from the sixth grade and 20 students from the seventh grade. The teacher was present during the application in both classes.

The work with the inverse operations is consolidated with the passing of the years and school levels. Therefore, we raise the following premises of this study: 1) reversible thinking will become more remarkable in the students of the seventh than in the students of the sixth grade of the school; 2) to advance in the levels of the activities proposed in the game, we need to recognize the inverse relationships between multiplication and division operations. Also, the use of mental calculation will be necessary to reach these premises.

The digital game was developed from the didactic sequence, validated by the scientific community, called *Fazendinha Matemática*[5] , and contains a series of analog games. It is a sequence that brings the storyline of a village, with several small farms, which suffers a natural catastrophe with devastating plantations and animals. For the recovery of the area, the residents decide to create an exchange system with the remaining animals and food, constructing an exchange table for this (figure 1). Exchanges are based on the binary system. This sequence has a series of text interpretation activities and mathematical content (especially arithmetic operations) and also involves exchanges with memory card games. The digital game is also called *Fazendinha Matemática*.

[5] This sequence by Tânia Gusmão has been developed since 1997 in public and private schools in Vitória da Conquista and aims to work on arithmetic operations in a playful and dynamic way. The sequence contains a set of activities and card games.

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Figure 1: swap table of the game *Fazendinha Matemática*

Source: projects linked to the game

The reader could ask himself why working the binary system. The initial idea is that working on a system that is shorter (in its digits) and easier to read, could result in a better system for students to understand ideas and concepts. In this context, we consider that "All arithmetic is supportive of the numbering system through what is expressed. [...] the different rules for obtaining a result in an operation are consistent with the system in which one works" (GÓMEZ, 1998, p.59)

For the current development of this research, we have a multidisciplinary team in which a professor from the UESB Cinema course, an IFBA Computer Science professor, three fellows from three UESB courses, Pedagogy, Cinema and Audiovisual, and Computer Science participate, plus the project coordinator, professor of the Mathematics Degree course. Although each member of the team has its particularities of knowledge in the areas in which they work, all collaborate in the scientific, philosophical, didactic, and pedagogical part of the game, so that it reaches the intended objectives. In previous years, this research also had the participation of other professors and alumni who also contributed to enrich the first version of the work, with its results published in the study of Dias, Gusmão, and Marques (2016).

Specifically, the team's work methodology was focused on the study and discussion of scientific texts related to the themes of digital games, task design, reversibility processes, and math content. Together with the studies, the game was being built on the digital platform, using the Unity platform in the first version. In this one, we worked very well on the plot of the original sequence, the history and the process of hunting, and the exchange of chicks. However, we noticed some limitations, mainly related to interactive implications that are part of the game design. "Game design is the process of developing an interactive application, which includes all aspects related to the design of a digital game" (NETO; ALVEZ, 2010, p.129).

In the second version, which we present here, we initially used the Gdevelop5 platform. However, difficulties with the management of this platform and the learning of a new programming language made us return to the initial Unity platform. Regarding the game art that involves the characters and scenarios, we used the Adobe Photoshop platform to design the game elements, which is a paid platform/software. The art was created in 2D, in the illustration models called Flat. This step required care to maintain an artistic unity in all drawings during the creative process, preserving the identity of each character, their characteristics, and history. We wanted the game with a more inviting and stimulating design, so we created its characters thought and executed in a way that all colors and scenarios followed the same chromatic scale and also drawing style. The idea was to preserve elements of the designs of the analog version but innovating them to have a digital appearance.

Below, we will show in detail the game phases produced in the second version, applied in the participating school.

The digital game and its educational intentions

We added five activities to the second version of the digital game.

The first phase consists of fourteen cards whose objective is to find the equivalent pairs, making simple and direct exchanges (figure 2), performing arithmetic multiplication and division operations mentally and at once. To do this, the player just has to consult the exchange table, clicking on the question mark icon in the upper corner of the screen (this table is also presented in the history of the *Fazendinha Matemática*) and list the objects that are equivalent in a direct relationship, respecting the hierarchy from the table (example: two chicks for a chicken; two chickens for a bag of corn, etc.).

Figura 2: Fase 1 do jogo, trocas simples e diretas

Source: projects linked to the game

To select the cards, at levels 1, 2, and 3, the player uses the mouse to move the cursor to the desired card and clicks the left button of the mouse to select it. Also, he can click the exit button to return to the main menu or the doubts (interrogation) button to access the table.

The second phase of the game consists of sixteen cards. In addition to finding the equivalent pairs, the player will perform more complex exchanges, which will correspond to the reverse exchanges (figure 3). In this phase, the player is hardly able to make immediate exchanges (following the hierarchy of the table, chicks for chickens, chickens for bags of corn...). Thus, he must perform mathematical operations of multiplication and division several times, so that the pair correspondent is found (example: eight chicks are equivalent to two bags of corn; sixteen bags of corn are equivalent to four sheep, etc.). This game allows the player to perform reversible thought processes. That is, to transform direct processes into inverse ones, in other words, to carry out “exchanges from front to back and back to front”.

Figure 3: Phase 2 of the game, exchanges with reverse reasoning

Source: projects linked to the game

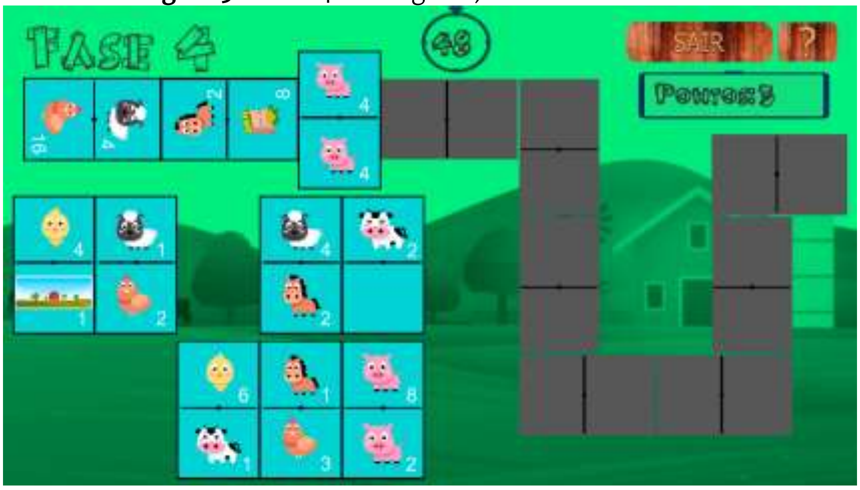
The third phase of the game consisted of sixteen cards and involves double representation, containing pieces with images and words (figure 4). The objective is the same, to find equivalent pairs of cards; however, in this phase, it is not worth finding pairs with the same representation (a word with a word; image with image), the player must find a word for each image. This phase requires the player to use reverse thinking.

Figure 4: Phase 2 of the game, exchanges with reverse reasoning

Source: projects linked to the game

The next two phases belong to a domino-style. Both phases will enable the use of reversible thinking. In phase 4 (figure 5), it should be played like the traditional domino pieces to fit the pieces through the equivalent exchanges (example: two cows for a plot of land; four chicks for two hens). At this stage, the game consists of ten domino pieces.

Figure 5: Phase 4 of the game, traditional dominoes

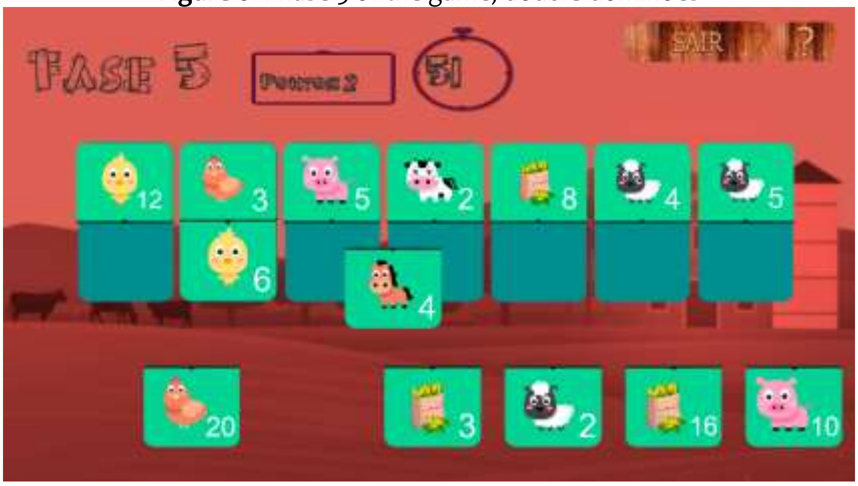


Source: projects linked to the game

To select the cards, at this level and the next level, the player uses the mouse to move the cursor to the desired card and click (and hold) the left button to drag it to the desired location where the card will be fixed if it is in its correct location. With the right mouse button, the player can rotate the card to be able to fit in certain places.

The fifth and final phase of this part of the game aims to assemble doubles (dowels) with equivalent exchanges (example: sixteen chickens and six chicks form a double). The domino game consists of fourteen pieces (half a domino piece) and a card containing the image of seven "blank" domino pieces in which the player must form the doubles and a place for the pieces to be assembled. (figure 6).

Figure 6: Phase 5 of the game, double-dominoes



Source: projects linked to the game

The educational objectives and intentions are repeated in the different phases on purpose. The idea is to vary the way of presenting the same content (mathematical ideas and concepts) with varying activities and providing different means of learning so that students learn, exercise, and apply skills. The more varied the activities to work on a mathematical concept or idea, the greater our chances of promoting the inclusion of our students, the more opportunities we will be giving them to learn. In our activities in particular, if they do not learn in one phase, it may be that in another they can and in case they do not succeed in the phases designed, we will have to design others or redesign the existing ones, aware that "the process of (re) design of tasks is important for the teacher, as it can be a starting point for his construction of knowledge and the development of reflection processes" (SOUSA et al 2019, p.451) for new applications in the classroom. This redesign process involves exercising and developing thinking flexibility.

We designed the new phases of the game to explore the relationships that multiplication and division operations have as an inverse of the other; to diagnose and develop ideas and concepts of inverse operations and; assess students' understanding of the concepts used (inverse operations, equivalence, and reversibility).

Experience Analysis and Discussion

In this item, we show a brief analysis and discussion of the experience carried out by applying the digital game to students of the sixth and seventh grades of an elementary school in a private school, seeking to articulate the bibliographic reference to our impressions.

The application of the game took place in the early hours of the morning on a Friday in March 2020. First, we applied it to students in the sixth grade and then to those in the seventh grade. The students listened carefully to the plot of the story of *Fazendinha Matemática* that was told by the project coordinator. At the same time, we presented some illustrative images of the story, including the exchange table. There was no time to make a tutorial of the game so that the story could be presented digitally and that is why the story was told through the project coordinator. Then, we informed the students that the game would be projected on the Data show and that they would be the players who would reverse each other in the phases of the game, with one or two players being able to manipulate/control the computer.

Illustration 1: classroom - 6th-grade students

Illustration 2: classroom - 7th-grade students

Source: projects linked to the game

In the sixth grade, many students with a lot of excited volunteered to play, so the teacher needed to control saying the order in which they would play. As soon as a student started playing, attentively many of the classmates who watched the projected game helped him by saying: “this is not the card... click on the other... the third... the one below... this one, That....”. Some students did not stay in their seats, not waiting for their turn to play, they got up to join the one who was controlling the computer.

The students' statements such as “this is not the letter are that other”, which helped those who controlled the game, demonstrated their different ways of thinking, representing the flexibility of thinking, but also manifestations of reversibility (INHELDER; PIAGET, 1958; KRUTETSKII 1976; WIERLEWSKI, 2005) since to find the letter that corresponded to the pair, it required a return of thought, sometimes far behind in the exchange table (for example, 1 horse equals 8 bags of corn).

In one of the phases of the game, two students - sixth grade - with autism asked to participate. Colleagues respectfully let them participate. One gave up and asked to leave the room because of the noise, the other ended the phase with the help of a colleague. The teacher asked if he liked the game and he said yes. In this particular episode, it was not possible to notice manifestations of reversible thinking.

As soon as the phases were over, the students asked to repeat and given the excitement and interest of them and also the time that remained of the class, we repeated the game.

In the seventh grade, fewer students volunteered to play. There was a certain shyness, especially from the girls. However, attention, concentration, and interest in the game was visible. The teacher intervened less. A maximum of two students was in front of the computer, controlling the game.

Illustration 3: student playing, 6th-grade

Illustration 4: student playing, 7th-grade

Source: projects linked to the game

It is important to highlight the presence and performance of the teacher who attentively and docilely gave voice to students who could not be heard by colleagues who were ahead and said: "Listen to what [name of the student] said .... [name of the student] wants to talk ... repeat what you said, [name of the student] ....". She still encouraged them to participate, saying: "look, how many chicks have the letter turned over?... 8 chicks exchange for what? ..." and contributed to the learning of the game. With this presence of mind, the teacher included everyone.

The teacher's questions stimulated students' mental calculations, causing them to think out loud, expressing their lines of reasoning. Some of them answered more quickly than others; some in a trial-and-error attempt attributed answers trying to certify; some thoughts aloud explicitly declared mastery of the rules of the exchange. When explaining the exchange rules, students showed the perception of the relationship that existed between multiplication and division operations (BELMONTE, 2003). When linking 1 (one) bag of corn for 8 (eight) chicks, they multiplied 2 x 1 step by step, trying to return to the starting point (reversible process) and once they were in the initial phase, they performed the division to check their calculations. This whole process was done mentally, out loud. They thought aloud, repeatedly making mistakes and correcting the mental calculations they presented and pausing in thought (wait a minute, I'm thinking, teacher), in which we started to perceive the flexibility in the students' thinking. We highlight the presence and the value of the error in the developed proposal, the observable error, that generated learning, and was part of the process as something natural and not feared (GUSMÃO; EMERIQUE, 2000). Students did not care about this variable.

All activities can also be seen from the point of view of mental calculation as an open problem (BELMONTE, 2003) given that students varied their calculation strategies, each with their particular strategy. The proposed activities forced students to seek, exercise, and apply calculation strategies, improving their knowledge of the numbering systems and the operations involved.

During the game, the classes did not consult the exchange table to clear up their doubts, they understood the rule from the explanation given initially by the coordinator when telling the story and the rest were learning by clicking letter by letter until finding the match. We reflect that we will have to create an objective and consistent tutorial, as we also evaluate that the students can be more agile in the plays and understanding of the rules.

Both classes evaluated the game positively, saying that it is very interesting and attractive and that it teaches math in a fun way, which confirms the game's potential for learning, as highlighted by some researchers (ALVES; BIACHIN, 2010; TELLES, 2015; RAMOS ET AL 2015). In this context, we consider that the variables interest and attraction contribute to favor flexible thinking, whenever they are variables that encourage the student to remain in the activity, not easily giving up on it, despite the challenges presented to them.

When asking them to suggest changes to the game to make it more attractive, the sixth graders said to change the soundtrack, adding more songs (there was a student who remembered the copyright and suggested that we create the soundtrack); to put more levels to spend more time playing; to make the game in 3D; that instead of increasing the size of the card at the moment of clicking the selection it should just darken the card.

The seventh graders also thought that the game should have more phases and changes in the soundtrack and it should signal when the time was running out, for example, increasing the speed of the music or changing the font color.

These observations show students' experience with technology, the mastery of elements of the language used by it, and how much they are used to digital games. This refers us to Alves and Biachin (2010, p.283) when they observe the importance of the act of playing, not only for the affective development but social and cognitive, because "the game is important, not only to stimulate the imagination in children but also to assist in the development of social and cognitive skills".

Although during the game we perceived demonstrations of the use of reverse thinking and flexibility by the students - when collectively they helped each other understand the rules of the game, making accounts out loud, presenting different ways of thinking, and struggling to think from back to front and finding the corresponding pairs of cards or set up the domino game - at the end of the game, we asked questions to obtain more evidence of the use of reverse reasoning and, in turn, of mental calculation. Thus, we asked how many chicks would be needed to be able to exchange for a bag of corn or a pig (smaller order in the exchange table). Sixth-grade students took longer to answer than seventh-grade students. When asked how many chicks would be needed to get a horse or a farm (higher-order in the exchange table), the sixth graders had more difficulties in answering. In both classes, some students used written calculus by taking pencils and paper to do the math, others used mental calculus by counting on their fingers and out loud. In general, the rules were more quickly assimilated by seventh graders and, as a result, they were more agile in finding the answers. The questions were intended to stimulate back and forth in thinking, to stimulate thinking to be flexible, a characteristic element of reflexivity, and to stimulate mental calculation strategies.

When asked about what math content they perceived in the game, the sixth graders answered division and multiplication, and seventh graders said multiples, division, multiplication, and power.

Below, we show the final score reached by the two classes at the end of the game. The illustrations also show the time spent (in seconds) in each phase and the final time (in minutes) spent by both classes, proving the greater agility of seventh-graders in understanding the rules of the game and, for example, consequence, in the ease of reversible thinking.

Illustration 5: final score, 6th-grade

Illustration 6: final score, 7th-grade

Source: projects linked to the game

Some problems, the so-called bugs, were detected during the application of the game: the increase or highlight of the selected card to find the corresponding pair was almost imperceptible to the students, causing them to be in doubt about which card they had selected; several letters were highlighted confusing; the last phase ended before the students put all the cards.

There were also problems with the game interface, we did not highlight or add color to the cards that did not match the desired pair (wrong cards) and increase the speed of the music when the time was running out.

Conclusions

When we understand that the work with inverse arithmetic operations is consolidated with the passing of years and school levels, we highlight that reversible thinking would become more noticeable in students in the seventh than in the sixth school year and that they would demonstrate such thinking when they perceive that to advance in the levels of the activities proposed in the game, we needed to recognize the relations that the multiplication and division operations have as an inverse of the other. Also, it would be necessary to use mental calculation. The assumptions of this study were verified in part because although the process of making "back-to-front" contests was notorious among both students of the participating classes, the time dedicated to the game and the number of activities carried out allowed us to observe the processes more carefully of mental calculation, due also to its speed, and the flexible thinking (changing strategies, making mistakes and correcting) that reversible processes to guarantee if they would be aware that one operation was the inverse of the other. That is, although they presented strategies for a return of thought to the point of origin when performing the required calculations in the activities, we cannot say that the students developed or have an understanding of the inverse operations.

On the other hand, whether or not consolidated the reversible thinking in these students, we evaluated that the game fulfilled its function of exploring the multiplication and division operations and, stimulating and developing indications of the concept of "reversibility" and exercising the mental calculation, whenever the activities were intended to lead students to reconstruct their mental process, encouraging them to change a direct thought to a reverse thought (KRUTETSKII, 1968 apud WIELEWSKI, 2005). Thus, we point out some characteristics that can contribute to the flexibility and reversibility of thinking, highlighted in the activities we develop, that is, in the digital game: to err and correct repeatedly, to think aloud, to change strategies, to pause thinking, to remain and insist on the activity, perform the mental calculation.

Through the game, we evaluated that the students' understanding of the inverse operations is in a direct relationship with the use made of them, that is, the work of perceiving the relationships between the multiplication and division operations does not seem to be routine, the primacy seems to be through direct thinking over the reverse.

The game provided interactions between students and, if used well and with educational intentions, it can give a range of return to the teacher's work, allowing to give voice to thoughts and ideas normally not heard and not evaluated.

We are more convinced that a way to help students understand the inverse operations using mental calculus is to insert them in rich environments with educational resources, as in the case with educational digital games, with the ability to arouse interest and develop concepts.

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